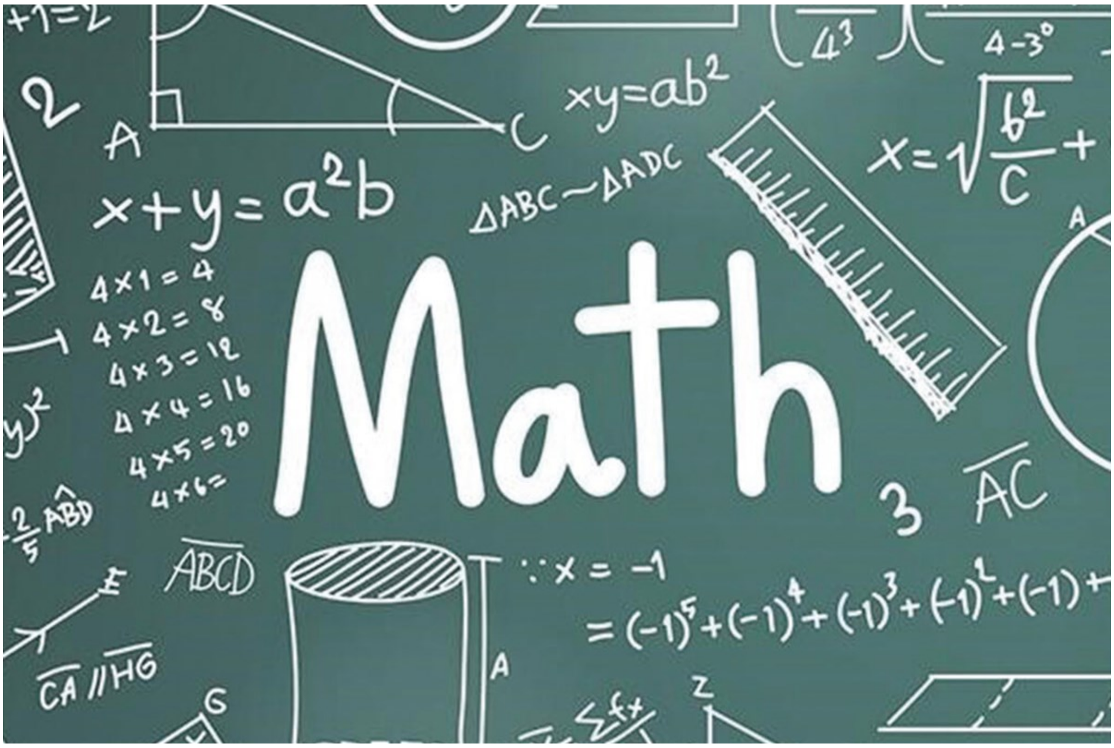

Mathematics

INTEGRAL CALCULUS

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Integra Calculus

Read this chapter carefully so that you can understand chapters like areas and volumes, multiple integrals etc. easily.

STANDARD RESULTS

Formula	Example
$\int x^n dx = \frac{x^{n+1}}{n+1}\ (n \neq 1)$	$\int x^{3/2} dx = \frac{x^{(3/2)+1}}{(3/2)+1} = \frac{2}{5}x^{5/2}$
$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a}$ where $n \neq -1$	$\int (2x+3)^5 dx = \frac{(2x+3)^6}{2x6} = \frac{(2x+3)^6}{12}$
$\int \frac{1}{x} dx = \log x$	
$\int \frac{dx}{(ax+b)} = \frac{1}{a}\log(ax+b)$	$\int \frac{1}{2x+7} dx = \frac{1}{2}\log(2x+7)$
$\int e^x dx = e^x$	
$\int e^{ax} dx = \frac{1}{a}e^{ax}$	
$\int a^x dx = \frac{a^x}{\log a}$	$\int 5^x dx = \frac{5^x}{\log 5}$
$\int a^{kx} dx = \frac{a^{kx}}{k \log a}$	
$\int \sin x.dx = -\cos x$	
$\int \cos x.dx = \sin x$	
$\int \tan x.dx = \log \sec x = -\log \cos x$	
$\int \cot x.dx = \log \sin x$	
$\int \sec x.dx = \log(\sec x + \tan x) = \log\left(\frac{\pi}{2} + \frac{x}{2}\right)$	
$\int \operatorname{cosec} x.dx = \log(\operatorname{cosec} x - \cot x) = \log . \tan\frac{x}{2}$	
$\int \sec^2 x.dx = \tan x$	

$\int \operatorname{cosec}^2 x.dx = -\cot x$	
$\int \sec x.\tan x.dx = \sec x$	
$\int \operatorname{cosec} x.\cot x.dx = -\operatorname{cosec} x$	
$\int \cosh x.dx = \sinh x$	
$\int \sinh dx = \cosh x$	
$\int af(x)dx = a\int f(x)dx$	$\int \sin 3x dx = -\frac{1}{3}\cos 3x$

Example

Integrate $\int \frac{dx}{\sin(x-a)\sin(x-b)}$

Solution

$$I = \int \frac{dx}{\sin(x-a)\sin(x-b)}$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin(b-a)}{\sin(x-a)(\sin(x-b)}dx$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin[(x-a)-(x-b)]}{\sin(x-a)\sin(x-b)}dx$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin(x-a)\cos(x-b)-\cos(x-a)\sin(x-b)}{\sin(x-a)\sin(x-b)}dx$$

$$= \frac{1}{\sin(b-a)}\left(\int \cot(x-b)dx - \int \cot(x-a)dx\right)$$

$$= \frac{1}{\sin(b-a)}\left(\log\sin(x-b) - \log(x-a)\right)$$

$$= \frac{1}{\sin(b-a)}\log\left(\frac{\sin(x-b)}{\sin(x-a)}\right)$$

Problem

Integrate

(i) $\int a^{5x}.dx$

Ans: $\frac{a^{5x}}{5\log a}$

(ii) $\int \cos^2 x.dx$

Ans: $\frac{x}{2}+\frac{\sin 2x}{4}$

(iii) $\int \sin x^0.dx$

Ans: $-\frac{180}{\pi}\cos x^0$

(iv) $\int \frac{\sin x}{\sin(x-a)}.dx$

Ans: tanx - secx

SOME IMPORTANT RESULTS TO REMEMBER

Formula	Example
$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) = -\cos^{-1}\left(\frac{x}{a}\right)$	$\int \frac{dx}{\sqrt{2-x^2}} = \sin^{-1}\frac{x}{\sqrt{2}}$
$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) = \log\left(\frac{x+\sqrt{x^2+a^2}}{a}\right)$	
$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) = \log\left(\frac{x+\sqrt{x^2-a^2}}{a}\right)$	
$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$	$\int \frac{dx}{(x+1)^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$
$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$	
$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$	
$\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$	
$\int \sqrt{x^2-a^2}.dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)$	
$\int \sqrt{x^2+a^2}.dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$	
$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x = -\operatorname{cosec}^{-1}x$	$\int \frac{dx}{(x+1)\sqrt{(x+1)^2-1}} = \text{sex}^{-1}(x+1)$

Example

Evaluate

(i) $\int \frac{dx}{x^2+2x+3}$

(ii) $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

Solution

(i) $\int \frac{dx}{x^2+2x+3} = \int \frac{dx}{(x^2+2x+1)+2} = \int \frac{dx}{(x+1)^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$

(ii) $\int \frac{dx}{(x+1)\sqrt{x^2+2x}} = \int \frac{dx}{(x+1)\sqrt{(x^2+2x+1)-1}} = \int \frac{dx}{(x+1)\sqrt{(x+1)^2-1}} = \sec^{-1}(x+1)$

Note

If the given expression is such that it can be easily transformed into some standard form, then transform it and then write the result directly as done in previous example.

AN IMPORTANT TYPE OF INTEGRAL

(i) $\int \frac{f'(x)}{f(x)}dx = \log f(x)$

(ii) $\int \{f(x)\}^n f'(x)dx = \frac{\{f(x)\}^{n+1}}{n+1} \quad (n \neq -1)$

Therefore, integral of a fraction whose numerator is exact differential coefficient of its denominator is equal to the logarithm of its denominator.

Integrals of tanx, cotx, secx and cosecx can be found from this relation as follows:

(a) $\int \tan x dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{-\sin x}{\cos x}dx = -\log\cos x = \log(\cos x)^{-1} = \log\sec x$

(b) $\int \cot x = \int \frac{\cos x}{\sin x} = \log\sin x$ [Numerator is diff coefficient of denominator]

(c) $\int \sec x dx = \int \frac{\sec(\sec x+\tan x)}{\sec x+\tan x}dx = \int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x} = \log(\sec x+\tan x)$

[Here Numerator is diff coefficient of denominator]

(d) $\int \operatorname{cosec} dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x+\cot x)}{\operatorname{cosec} x-\cot x}dx = \int \frac{-\cot x\operatorname{cosec} x+\operatorname{cosec}^2 x}{\operatorname{cosec} x-\cot x}dx$

$$= \log(\operatorname{cosec} x - \cot x)$$

(e) $\int \operatorname{cosec} x\,dx = \int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}}$

Dividing numerator and denominator by $\cos^2\left(\frac{x}{2}\right)$, we have,

$$\int \operatorname{cosec} x\,dx = \int \frac{\sec^2\left(\frac{x}{2}\right)dx}{2\tan\left(\frac{x}{2}\right)} = \int \frac{\frac{1}{2}\sec^2(x/2)}{\tan\left(\frac{x}{2}\right)}dx$$

Here numerator $\left(\frac{1}{2}\sec^2\frac{x}{2}\right)$ is the exact differential coefficient of the denominator $\left(\tan\frac{x}{2}\right)$.

$\therefore \qquad \int \operatorname{cosec} x\,dx = \log\tan\frac{x}{2}$

(f) $\int \sec x\,dx = \int \operatorname{cosec}\left(\frac{\pi}{2}+x\right)dx = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) \qquad \left(\text{because} \int \operatorname{cosec} x\,dx = \log\tan\frac{x}{2}\right)$

Example

Integrate

(i) $\int \frac{e^{\tan^{-1}x}}{1+x^2}dx$

(ii) $\int \frac{x+(\cos^{-1}3x)^2}{\sqrt{1-9x^2}}dx$

Solution

(i) $\qquad I = \int \frac{e^{\tan^{-1}x}}{1+x^2}dx$

Put $\tan^{-1}x = t \rightarrow \frac{1}{1+x^2}dx = dt$

$\therefore \qquad I = \int e^t dt = e^t = e^{\tan^{-1}x}$

(ii) $\qquad I = \int \frac{x}{\sqrt{1-9x^2}}dx + \frac{1}{3}\int \frac{(\cos^{-1}3x)^2.3dx}{\sqrt{1-9x^2}}$

$$= -\frac{1}{18}\int \frac{-18x}{\sqrt{1-9x^2}}dx + \frac{1}{3}\int \frac{(\cos^{-1}3x)^2.3dx}{\sqrt{1-9x^2}}$$

Put $1 - 9x^2 = t$ and $\cos^{-1}3x = u$

$$-18x\,dx = dt \text{ and } \frac{-3}{\sqrt{1-9x^2}}dx = du$$

$$I = -\frac{1}{18}\int\frac{dt}{\sqrt{t}} - \frac{1}{3}\int u^2 du = -\frac{1}{18}\int t^{-1/2}dt - \frac{1}{3}\frac{u^3}{3}$$

$$= -\frac{1}{9}\sqrt{t} - \frac{u^3}{9} = -\frac{1}{9}\sqrt{1-9x^2} - \frac{(\cos^{-1}3x)^3}{9}$$

Note

The above type of problems fall under the method of integration by substitution. In which a change in the variable of integration often reduces an integral to one of the standard forms. Thus, the given integral can be evaluated with the help of the substitution of a new variable.

INTEGRATION BY PARTS

If $f_1(x)$ and $f_2(x)$ are two functions of "x", then the integral of their product may be given as follows:

Integral of the product of two functions

First function x integral of second - Integral of[Diff. coeff. of first x Integral of second]i.e.

$$\int f_1(x).f_2(x).dx = f_1(x).\int f_2(x)dx - \int[f'_1(x).\int f_2(x)dx]dx$$

Example

$$\int x\sin x dx = (x)\int \sin x dx - \int\left(\frac{d}{dx}(x).\int \sin x dx\right)dx$$

$$= x(-\cos x) - \int 1.(-\cos x)dx$$

$$= -x\cos x + \sin x$$

Integration of the function with the help of above rule is called integration by parts.

NOTE

- If the integrand is a product of the positive integral power of x and either a exponential or logarithm or a trigonometric function, the function which is easy to integrate should be taken as second function.
- If there is only function whose integral is not known, multiply it by one and take one as the second function.
- If the integrals of both the functions are known, the function which is easy to integrate is taken as the second function.
- Rule can be applied repeatedly to evaluate the integral.

Example

Integrate

(i) $\int \frac{\log(1+x)}{(2x+1)^2} dx$

(ii) $\int \tan^{-1} x dx$

Solution

(i) $I = \int \frac{\log(1+x)}{(2x+1)^2}.dx$

$$= \log(1+x).\frac{(2x+1)^{-1}}{-2} + \int \frac{1}{1+x}.\frac{1}{-2(2x+1)} dx$$

$$= -\frac{\log(1+x)}{2(2x+1)} + \frac{1}{2}\int \frac{dx}{(x+1)(2x+1)}$$

$$= -\frac{\log(1+x)}{2(2x+1)} + \frac{1}{2}\int \left(\frac{-1}{x+1} + \frac{2}{2x+1}\right) dx$$

$$= -\frac{\log(1+x)}{2(2x+1)} - \frac{1}{2}\log(x+1) + \frac{1}{2}\log(2x+1)$$

$$= -\frac{1}{2}\left(\frac{\log(x+1)}{(2x+1)} + \frac{\log(x+1)}{(2x+1)}\right)$$

(ii) $\int \tan^{-1} x dx = \int \tan^{-1} x.1.dx$

$$= \tan^{-1} x.x - \int \frac{1}{1+x^2}.x.dx$$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2} dx$$

$$= x\tan^{-1} x - \frac{1}{2}\log(1+x^2)$$

Problem

Evaluate

(i) $\int x.\sin^{-1} x dx$

Ans: $\frac{1}{2}\left[x^2 \sin^{-1} x + \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1} x\right]$

(ii) $\int \frac{x + \sin x}{1 + \cos x} dx$

Ans: $x \tan(x/2)$

Hint: $\int \frac{x + \sin x}{1 + \cos x} dx = \int \frac{x + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2(x/2)} = \frac{1}{2}\int x \sec^2 \frac{x}{2} dx + \int \tan\frac{x}{2} dx$

Integrate the first integral on RHS by parts.

THREE STANDARD FORMS

To integrate

(i) $\int \sqrt{a^2 - x^2} dx$

(ii) $\int \sqrt{a^2 - x^2}$

(iii) $\sqrt{x^2 - a^2}$

Solution

(i) $\int \sqrt{a^2 - x^2} dx = \int \sqrt{a^2 + x^2}.1.dx$

Integrating by parts taking $\sqrt{a^2 - x^2}$ as the first function

$$= \sqrt{a^2 - x^2}.x - \int \frac{1}{2}(a^2 - x^2)^{1/2}(-2x).x.dx$$

$$= x\sqrt{a^2 - x^2} + \int \frac{x^2}{\sqrt{a^2 - x^2}}.dx$$

$$= x\sqrt{a^2 - x^2} + \int \frac{-(a^2 - x^2) + a^2}{\sqrt{a^2 - x^2}}.dx$$

$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}.dx + a^2 \int \frac{1}{\sqrt{a^2 - x^2}}.dx$$

$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2} dx + a^2 \sin^{-1}\frac{x}{a}$$

$\therefore \quad 2\int \sqrt{a^2 - x^2}.dx = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}$ [Transposing]

$\therefore \quad \int \sqrt{a^2 - x^2}.dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$ [Result]

Other two integrals, i.e. $\int \sqrt{a^2 + x^2}.dx$ and $\int \sqrt{x^2 - a^2} dx$ may be similarly found out. The results are as follows:

(ii) $\int \sqrt{a^2+x^2} = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$ [Form I]

$= \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\log\frac{x+\sqrt{a^2+x^2}}{a}$ [Form II]

$\left[\because \sin^{-1}\frac{x}{a} = \log\frac{x+\sqrt{a^2+x^2}}{a}\right]$

(iii) $\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$ [Form I]

$= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\frac{x+\sqrt{x^2-a^2}}{a}$ [Form II]

$\left[\because \cos^{-1}\frac{x}{a} = \log\frac{x+\sqrt{x^2-a^2}}{a}\right]$

Problem

Evaluate $\int \frac{x^2-2x+3}{\sqrt{1-x^2}}dx$

Ans: $\frac{7}{2}\sin^{-1}x + 2\sqrt{1-x^2} - \frac{x\sqrt{1-x^2}}{2}$

Hint: $x^2-2x+3 = 4-2x-(1-x^2)$

INTEGRATION OF RATIONAL ALGEBRAIC FUNCTIONS

To evaluate integrals of $\int \frac{1}{ax^2+bx+c}$ type, first complete the square in the denominator. For this, first make the coefficient of x^2 unity. Then add and subtract square of half the coefficient of "x", i.e. put the denominator in the form $a\{(x+\alpha)^2 \pm \beta^2\}$ and then integrate.

Three standard forms which are used in such types of problems are:

(i) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}$

(ii) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}$

(iii) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

Example

Solve $\int \frac{dx}{3x^2+6x+21}$

Solution

$$I = \frac{1}{3}\int \frac{dx}{x^2+2x+7}$$

$$= \frac{1}{3}\int \frac{dx}{(x^2+2x+1)+7-1}$$

$$= \frac{1}{3}\int \frac{dx}{(x+1)^2+(\sqrt{6})^2}$$

$$= \frac{1}{3\sqrt{6}}\tan^{-1}\left(\frac{x+1}{\sqrt{6}}\right)$$

Note

For integration of the type $\int \frac{px+q}{ax^2+bx+c}$, break the given fraction into two fractions such that in one of them numerator is the diff. coeff. of the denominator and in the other numerator is only a constant. This would become quite clear from the example given below.

Example

Evaluate $\int \frac{5x-2}{1+2x+3x^2}.dx$

Solution

$$I = \int \frac{(5/6)(6x+2)-2-\frac{5}{2}}{3x^2+2x+1}dx = \int \frac{(5/6)(6x+2)-\frac{11}{3}}{3x^2+2x+1}.dx$$

$$= \frac{5}{6}\int \frac{6x+2}{3x^2+2x+1} - \frac{11}{3}\int \frac{dx}{x^2+(2/3)x+(1/3)}$$

$$= \frac{5}{6}\log(3x^2+2x+1) - \frac{11}{9}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2+\frac{2}{9}}$$

$$= \frac{5}{6}\log(3x^2+2x+1) - \frac{11}{9}.9\int \frac{dx}{(3x+1)^2+(\sqrt{2})^2}$$

$$= \frac{5}{6}\log(3x^2+2x+1) - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)$$

$$= \frac{5}{6}\log(3x^2+2x+1) - \frac{11\sqrt{2}}{6}\tan^{-1}[(3x+1)/\sqrt{2}]$$

INTEGRATION BY PARTIAL FRACTIONS

Case I

When factors of denominator are all of the first degree and each occurring once only, then assume a fraction of the form $\frac{A}{ax+b}$ for each factor of the form ax+b, as explained in following example.

Example

Evaluate $\int \frac{x+1}{x(x-1)(x+2)}dx$

Solution

Let $$\frac{x+1}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

Multiplying both sides by L.C.M., we have

$$x+1 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

Now Put x = 0,1,-2

When x = 0, A = -1/2

When x = 1, B = 2/3

When x = -2, C = -1/6

$$\therefore \quad \frac{x+1}{x(x-1)(x+2)} = -\frac{1}{2x} + \frac{2}{3(x-1)} - \frac{1}{6(x+2)}$$

$$\therefore \quad \int \frac{x+1}{x(x-1)(x+2)}.dx = -\frac{1}{2}\int \frac{dx}{x} + \frac{2}{3}\int \frac{dx}{x-1} - \frac{1}{6}\int \frac{dx}{x+2}$$

Case II

When factors of denominator are all of first degree but some factors of the form ax+b are repeated "r" times in the denominator, then partial fractions are

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + ...\frac{A_r}{(ax+b)^r}$$

The above details will be clear from the following example.

Example

Evaluate $\int \frac{x^2+x+1}{(x-1)^2(x-2)}dx$

Solution

Let $\dfrac{x^2+x+1}{(x-1)^2(x-2)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x-2}$

Multiplying both sides by L.C.M., we have

$$x^2+x+1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2 \; .... \qquad (1)$$

Putting x = 1,2 we get

B = -3 and C = 7

Now equating coeff. of x^2 on both sides of (1), we get

A + C = 1 or A + 7 = 1 or A = -6

$$\therefore \qquad I = -6\int \frac{dx}{x-1} - 3\int \frac{dx}{(x-1)^2} + 7\int \frac{dx}{x-2}$$

$$= -6\log(x-1) - 3\left(\frac{-1}{x-1}\right) + 7\log(x-2)$$

$$= -6\log(x-1) + \frac{3}{x-1} + 7\log(x-2)$$

Case III

Whenever the denominator contains some quadratic factors which can not be factorised further, then for each factor of the form $ax^2 + bx + c$ in the denominator, we suppose a fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$, which would become clear from following example.

Example

Evaluate $\displaystyle\int \frac{x}{(x-1)(x^2+4)} dx$

Solution

Let $\dfrac{x}{(x-1)(x^2+4)} dx = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+4}$

Multiplying both sides by L.C.M., we get

$$x = A(x^2+4) + (Bx+C)(x-1)$$

Putting, x = 1

We get A = 1/5

Equating the coefficient of x^2 and constant terms on both sides, we get

$0 = A + B \rightarrow B = -1/5$

and $0 = 4A - C \to C = 4A = 4/5$

$$\therefore \quad \frac{x}{(x-1)(x^2+4)} = \frac{1}{5(x-1)} - \frac{x-4}{5(x^2+4)}$$

$$\therefore \quad \int \frac{x}{(x-1)(x^2+4)}dx = \frac{1}{5}\int \frac{dx}{(x-1)} - \frac{1}{5}\int \frac{x-4}{x^2+4}dx$$

$$= \frac{1}{5}\log(x-1) - \frac{1}{5}\int \frac{x}{x^2+4}dx + \frac{4}{5}\int \frac{1}{x^2+4}dx$$

$$= \frac{1}{5}\log(x-1) - \frac{1}{10}\int \frac{2x}{x^2+4}.dx + \frac{4}{5}\int \frac{1}{x^2+4}$$

$$= \frac{1}{5}\log(x-1) - \frac{1}{10}\log(x^2+4) + \frac{4}{5}\tan^{-1}\frac{x}{2}$$

Problem

Evaluate

(i) $\int \frac{3x.dx}{(x-1)(x-2)(x-3)}$

Ans: $\frac{3}{2}\log(x-1) - 6\log(x-2) + \frac{9}{2}\log(x-3)$

(ii) $\int \frac{x^2dx}{x^2+7x+10}$

Ans: $x + \frac{4}{3}\log(x+2) - \frac{25}{3}\log(x+5)$

(iii) $\int \frac{3x+2}{(x+1)^2(x-2)}$

Ans: $-\frac{1}{3}.\frac{1}{x+1} + \frac{8}{9}\log\frac{x-2}{x+1}$

Note

Sometimes, the working can be considerably reduced by using first a suitable substitution.

- If the problem is a function of e^x, put $e^x = y$
- If either numerator or denominator contains only odd powers of "x" while all the remaining terms are of even powers of "x", then put $x^2 = y$.
- If integral is of the form $\int \frac{dx}{x(x^n+1)}$, put $x^n = y$.

The above three points would be cleared in following three examples one by one.

Example

Evaluate

(i) $\int \frac{dx}{e^x - 1}$

(ii) $\int \frac{2x}{(x^2+1)(x^2+3)} dx$

Solution

(i) $I = \int \frac{dx}{e^x - 1}$

Put $e^x = y \therefore e^x dx = dy$

$\therefore \quad I = \int \frac{dx}{e^x - 1} = \int \frac{e^x dx}{e^x(e^x - 1)} = \int \frac{dy}{y(1-y)}$

Let $\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}$

$1 = A(1 - y) + By$

Putting y = 0, 1. We get A = 1 and B = 1

$\therefore \quad I = \int \left[\frac{1}{y} + \frac{1}{1-y}\right] dy = \log y - \log(1-y) = \log\left[\frac{y}{1-y}\right] = \log\left[\frac{e^x}{1-e^x}\right]$

(ii) $I = \int \frac{2x}{(x^2+1)(x^2+3)} dx$

Put $x^2 = y$ $\qquad \therefore 2x.dx = dy$

$\therefore \quad I = \int \frac{dy}{(y+1)(y+3)}$

Let $\frac{1}{(y+1)(y+3)} = \frac{A}{y+1} + \frac{B}{y+3}$

Solving A = 1/2 and B = -1/2

$\therefore \quad I = \frac{1}{2}\int \left[\frac{1}{y+1} - \frac{1}{y+3}\right] dy = \frac{1}{2}\left[\log(y+1) - \log(y+3)\right]$

$= \log\left[\frac{y+1}{y+3}\right] = \frac{1}{2}\log\left[\frac{x^2+1}{x^2+3}\right]$

(iii) $I = \int \frac{dx}{x(x^4+1)}$

Put $x^4 = y \rightarrow 4x^3dx = dy$ or $x^3\, dx = dy/4$

$$I = \int \frac{x^3dx}{x^4(x^4+1)} = \int \frac{dy}{4y(y+1)} = \frac{1}{4}\int \frac{dy}{y(y+1)}$$

Let $$\frac{1}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1}$$

Solving A = 1 and B = -1

$$\therefore \quad \int \left[\frac{1}{y} - \frac{1}{y+1}\right]dy = \frac{1}{y}\log\left[\frac{y}{y+1}\right] = \frac{1}{y}\log\left[\frac{x^4}{x^4+1}\right]$$

SOME IMPORTANT TYPES OF INTEGRALS

In the examples given in following article, we would see how to solve the integrals of the type.

(i) $$\int \frac{x^2+1}{x^4+kx^2+1}dx$$

(ii) $$\int \frac{x^2-1}{x^4+kx^2+1}dx$$

(iii) $$\int \frac{ax^2+b}{x^4+kx^2+1}dx$$

Example

Solve

(i) $$\int \frac{x^2+1}{x^4+1}dx$$

(ii) $$\int \frac{x^2-1}{x^4+x^2+1}dx$$

Solution

(i) $$I = \int \frac{x^2+1}{x^4+1}dx$$

Dividing numerator and denominator by x^2, we get

$$I = \int \frac{\left(1+\frac{1}{x^2}\right)}{x^2+\frac{1}{x^2}}dx = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2+2}dx$$

Put $x - \frac{1}{x} = y \quad \therefore \left(1+\frac{1}{x^2}\right)dx = dy$

$$\therefore \qquad I = \int \frac{dx}{y^2 + 2} = \int \frac{dx}{y^2 + (\sqrt{2})^2} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{y}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left[\frac{\left(x - \frac{1}{x}\right)}{\sqrt{2}}\right] = \frac{1}{\sqrt{2}} \tan^{-1}\left[\frac{x^2 - 1}{x\sqrt{2}}\right]$$

(ii) $I = \int \frac{x^2 - 1}{x^4 + 1} dx$

Dividing numerator and denominator by x^2, we get

$$I = \int \frac{\left(1 - \frac{1}{x^2}\right)}{x^2 + \frac{1}{x^2}} dx = \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 - 2} dx$$

Put $x + \frac{1}{x} = y \quad \therefore \left(1 - \frac{1}{x^2}\right) dx = dy$

$$I = \int \frac{dy}{y^2 - (\sqrt{2})^2} = \frac{1}{2\sqrt{2}} \log \frac{y - \sqrt{2}}{y + \sqrt{2}} \qquad \left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x - 1}{x + a}\right]$$

$$\therefore \qquad I = \frac{1}{2\sqrt{2}} \log\left(\frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}}\right) = \frac{1}{2\sqrt{2}} \log\left(\frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1}\right)$$

INTEGRATION OF IRRATIONAL ALGEBRAIC FUNCTIONS

To integrate a rational function of x and a linear surd of the type $(ax+b)^{1/n}$ where n is some positive integer, put $(ax+b)^{1/n} = t$.

To integrate a rational function of "x" and a surd of the type $\left[\frac{ax + b}{cx + d}\right]^{1/n}$, put $\left[\frac{ax + b}{cx + d}\right]^{1/n} = t$

Example

Evaluate

(i) $\int_1^2 \sqrt{\frac{x - 1}{2 - x}} dx$

(ii) $\int \frac{x^{1/4}}{1 + \sqrt{x}} dx$

Solution

(i) $I = \int_1^2 \sqrt{\frac{x-1}{2-x}}dx$

Let $\sqrt{\frac{x-1}{2-x}} = t$ or $\frac{x-1}{2-x} = t^2$

Hence $x = [(1+2t^2)/(1+t^2)]$

Hence $dx = [(1+t^2).4t - (1+2t^2).2t/(1-t^2)^2]dt$

$$\left(\frac{4t-4t^3-2t-4t^3}{(1+t^2)^2}\right)dt = \frac{2tdt}{(1+t^2)^2}$$

Hence $I = \int_1^2 \sqrt{\frac{x-1}{2-x}}dx = \int_0^\infty \frac{2t^2}{(1+t^2)^2}dt$

[because when x = 1 then t = 0 and when x = 2 then t = ∞]

Let $t = \tan\theta$ $\therefore$ $dt = \sec^2\theta\, d\theta$

$\therefore$ $I = 2\int_0^{\pi/2} \frac{\tan^2\theta \sec^2\theta}{\sec^4\theta}d\theta = 2\int_0^{\pi/2}\sin^2\theta d\theta$

[because when t= 0 then θ = 0 and when t = ∞ then θ = π/2]

$$= \int_0^{\pi/2}[1-\cos 2\theta]d\theta = \left|\theta - \frac{\sin 2\theta}{2}\right|_0^{\pi/2} = \frac{\pi}{2}$$

(ii) $I = \int \frac{x^{1/4}}{1+\sqrt{x}}dx$

Let $x^{1/4} = t$ Hence $x = t^4$ or $dx = 4t^3dt$

$\therefore$ $\int \frac{t}{1+t^2}.4t^2.dt = 4\int \frac{t^4dt}{t^2+1} = \int\left(t^2-1+\frac{1}{t^2+1}\right)dt = 4\left(\frac{t^3}{3} - t + \tan^{-1} t\right)$

=

$$\frac{4}{3}t^3 - 4t + 4\tan^{-1} t = \frac{4}{3}(x^{1/4}) - 4(x^{1/4}) + 4\tan^{-1}(x^{1/4})$$

Note

To integrate $\sqrt{ax^2+bx+c}$ or $1/\sqrt{ax^2+bx+c}$, complete the square and bring the expression in any of the following standard forms like $\int\sqrt{a^2+x^2}dx, \int\sqrt{a^2-x^2}dx, \int\sqrt{x^2-a^2}, \int\frac{1}{\sqrt{x^2-a^2}}dx, \int\frac{1}{\sqrt{a^2+x^2}}dx, \int\frac{1}{\sqrt{a^2-x^2}}dx$

Example

Solve

(i) $\int \sqrt{x(1-x)}dx$

(ii) $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

Solution

(i) $$I = \int \sqrt{x-x^2}dx = \int \sqrt{\frac{1}{4}-\left(x^2-x+\frac{1}{4}\right)}dx = \int \sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\,dx$$

This is in the standard form of $\int \sqrt{a^2-x^2}dx$

$$\therefore I = \left(\frac{2x-1}{4}\right)\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}+\frac{1}{8}\sin^{-1}(2x-1)$$

(ii) $$I = \int \frac{dx}{\sqrt{x\beta-x^2-\alpha\beta+x\alpha}} = \int \frac{dx}{\sqrt{-x^2+(\alpha+\beta)x-\alpha\beta}}$$

$$= \int \frac{dx}{\sqrt{\left(\frac{\alpha+\beta}{2}\right)^2-\alpha\beta-\left(x^2-(\alpha+\beta)x+\left(\frac{\alpha+\beta}{2}\right)^2\right)}} = \int \frac{dx}{\sqrt{\left(\frac{\alpha-\beta}{2}\right)^2-\left(x-\frac{\alpha+\beta}{2}\right)^2}}$$

This is in standard form of $\int \frac{dx}{\sqrt{a^2-x^2}}$

$$\therefore \quad I = \sin^{-1}\left(\frac{x-\frac{\alpha+\beta}{2}}{\frac{\alpha-\beta}{2}}\right)$$

Note

To integrate the expressions like $\int \frac{1}{X\sqrt{Y}}dx$,where X and Y are functions of "x", we must perform following substitutions:

(i) If X and Y both are of first degree, i.e. linear, then put $\sqrt{Y} = t$.
(ii) If X is linear, Y quadratic, then put $X = 1/t$
(iii) If X is quadratic, Y linear, then put $\sqrt{Y} = t$
(iv) If X and Y are both quadratic, put $\sqrt{(Y/X)} = t$

Example

Solve

(i) $\int \frac{dx}{(x+2)\sqrt{x+3}}$

(ii) $\int \frac{dx}{x^2\sqrt{4-x^2}}$

Solution

(i) $I = \int \frac{dx}{(x+2)\sqrt{x+3}}$

Here (x+2) and (x+3) are both linear, hence

Put $\sqrt{x+3} = t$

$\therefore$ $x + 3 = t^2$ or $dx = 2t.dt$

$\therefore\ I = \int \frac{2tdt}{(t^2-3+2)t} = \int \frac{2dt}{(t^2-1)}$

$= \int \frac{2dt}{(t+1)(t-1)} = \int \left(\frac{1}{t-1} - \frac{1}{t+1}\right)dt$

$=$

$\log(t-1) - \log(t+1) = \log\left(\frac{t-1}{t+1}\right) = \log\frac{\sqrt{x+3}-1}{\sqrt{x+3}+1}$

(ii) $I = \int \frac{dx}{x^2\sqrt{4-x^2}}$

Here, x^2 and $(4 - x^2)$ both are quadratic, hence

Let $\sqrt{\frac{4-x^2}{x^2}} = t$ or $\frac{4-x^2}{x^2} = t^2$

$\therefore$ $t^2x^2 = 4 - x^2$ or $t^2x^2 + x^2 = 4$

or $x^2 = \frac{4}{1+t^2}$

$\therefore$ $x = \frac{2}{\sqrt{t^2+1}} = 2(t^2+1)^{-1/2}$

$\therefore$ $dx = 2\left(-\frac{1}{2}\right)\left(t^2+1\right)^{-3/2} x.2t = \frac{-2t}{(t^2+1)^{3/2}}$

Now, $4 - x^2 = 4 - \frac{4}{t^2+1} = \frac{4t^2+4-4}{t^2+1} = \frac{4t^2}{t^2+1}$

$$\therefore \quad I = \int -\frac{2t}{(t^2+1)^{3/2}} \times \frac{dt}{\frac{4}{t^2+1}.\frac{2t}{\sqrt{t^2+1}}} = -\frac{1}{4}\int dt = -\frac{1}{4}t = -\frac{1}{4}\left(\frac{\sqrt{4-x^2}}{x}\right)$$

INTEGRATION OF TRIGONOMETRIC FUNCTIONS

To evaluate $\int \sin^m x \cos^n x dx$

(i) If "m" is odd, put cosx = t. If "n" is odd put sinx = t. If both "m" and "n" are odd, then put either cosx or sinx equal to "t", and integrate.
(ii) When (m + n) is a negative even integer, then express the given integral as the product of powers of tanxsecx and put tanx = t.
(iii) If "m" and "n" are small even integers then express $\sin^m x\cos^n x$ in terms of multiples of angles.

Example

Solve $I = \int \sin^2 x \cos^2 x.dx$

Solution

Here power of cosx is 7, which is odd.

$\therefore$ Let sinx = t → cosx.dx = dt

$$\therefore \quad I = \int \sin^2 x.\cos^6 x.\cos x dx = \int \sin^2 x(1-\sin^2 x)^3 \cos x dx$$

$$= \int t^2(1-t^2)^3 dt = \int t^2(1-3t^2+3t^4+t^6)dt$$

$$= \int (t^3 - 3t^4 + 3t^6 - t^8)dt = \frac{t^3}{3} - \frac{3t^5}{5} + \frac{3t^7}{7} - \frac{t^9}{9}$$

$$= \frac{1}{3}\sin^3 x - \frac{3}{5}\sin^5 x + \frac{3}{7}\sin^7 x - \frac{1}{9}\sin^9 x$$

Note

For integrals of the types (i) $\int \frac{dx}{a+b\sin x}$ (ii) $\int \frac{dx}{a+b\cos x}$ (iii) $\int \frac{dx}{a+b\sin x+\cos x}$

First put $\sin x = \frac{2\tan(x/2)}{1+\tan^2(x/2)}$ and $\cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$

and then put tan(x/2) = t.

Example

Solve

(i) $\int \frac{dx}{a+b\sin x}$

(ii) $\int \frac{dx}{2+\sin 2x}$

Solution

(i) $$I = \int \frac{dx}{a+b\sin x} = \int \frac{dx}{a+b\left[\frac{2\tan(x/2)}{1+\tan^2(x/2)}\right]}$$

Now, Let tan(x/2) = t or $\frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx = dt$

$$\therefore \quad dx = \frac{2dt}{\sec^2(x/2)} = \frac{2dt}{1+\tan^2(x/2)} = \frac{2dt}{1+t^2}$$

$$\therefore \quad I = \int \frac{dx}{a+\frac{2bt}{1+t^2}} = \int \frac{\frac{2dt}{1+t^2}}{\frac{at^2+2bt+a}{1+t^2}}$$

$$= 2\int \frac{dt}{at^2+2bt+a} = \frac{2}{a}\int \frac{dt}{t^2+\frac{2b}{a}t+1}$$

$$= \frac{2}{a}\int \frac{dt}{\left(t^2+\frac{2b}{a}t+\frac{b^2}{a^2}\right)+1-\frac{b^2}{a^2}} = \frac{2}{a}\int \frac{dt}{\left(t+\frac{b}{a}\right)^2+\frac{a^2-b^2}{a^2}}$$

Now, there are three possibilities.

Case I: a > b

$$\therefore \quad I = \frac{2}{a}\int \frac{dt}{\left(t+\frac{b}{a}\right)^2+\left(\frac{\sqrt{a^2-b^2}}{a}\right)^2}$$

$$= \frac{2}{a}.\frac{a}{\sqrt{a^2-b^2}}\tan^{-1}\left(\frac{at+b}{\sqrt{a^2-b^2}}\right)$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\frac{a\tan(x/2)+b}{\sqrt{a^2-b^2}}\right)$$

Case II: a < b or b > a

$$\therefore \qquad I = \frac{2}{a}\int \frac{dt}{\left(t+\frac{b}{a}\right)^2 + \frac{b^2-a^2}{a^2}}$$

$$= \frac{2}{a}\int \frac{dt}{\left(t+\frac{b}{a}\right)^2 + \left(\frac{\sqrt{b^2-a^2}}{a}\right)^2}$$

$$= \frac{2}{a}.\frac{a}{2\sqrt{b^2-a^2}}\log\left(\frac{t+\frac{b}{a}-\frac{\sqrt{b^2-a^2}}{a}}{t+\frac{b}{a}+\frac{\sqrt{b^2-a^2}}{a}}\right)$$

$$= \frac{1}{\sqrt{b^2-a^2}}\log\left(\frac{a\tan(x/2)+b-\sqrt{b^2-a^2}}{a\tan(x/2)+b+\sqrt{b^2-a^2}}\right)$$

Case III: a = b

$$\therefore \qquad \frac{2}{a}\int\frac{dt}{(t+1)^2} = \frac{2}{a}\int (t-1)^{-2}.dt = \frac{2}{a}.\frac{(t+1)^{-1}}{-1} = -\frac{2}{a(t+1)} = -\frac{2}{a\left(\tan(x/2)+1\right)}$$

(ii) $$I = \int\frac{dx}{2+\sin 2x} = \int\frac{dx}{2+\left(\frac{2\tan x}{1+\tan^2 x}\right)}$$

Let, tanx = t or $\sec^2 x.dx = dt$

$$\therefore \qquad I = \int\frac{\frac{dt}{1+t^2}}{\left(\frac{2+2t^2+2t}{1+t^2}\right)} = \frac{1}{2}\int\frac{dt}{t^2+t+1}$$

$$= \frac{1}{2}\int\frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{2}.\frac{2}{\sqrt{3}}.\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}}.\tan^{-1}\left(\frac{2\tan x+1}{\sqrt{3}}\right)$$

Note

For integrals of the types

(i) $\int \frac{dx}{a+b\sin^2 x}$

(ii) $\int \frac{dx}{a+b\cos^2 x}$

(iii) $\int \frac{dx}{a\cos^2 x+b\sin^2 x}$

(iv) $\int \frac{dx}{(a\cos x+b\sin x)^2}$

Divide numerator and denominator by $\cos^2 x$, and put $\tan x = t$.

Example

Solve $\int \frac{dx}{1+3\sin^2 x}$

Solution

$$I = \int \frac{dx}{\sin^2 x+\cos^2 x+3\sin^2 x} = \int \frac{dx}{4\sin^2 x+\cos^2 x}$$

Now, dividing numerator and denominator by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x dx}{4\tan^2 x+1}$$

Put $2\tan x = t \;\therefore\; 2\sec^2 x dx = dt$

$\therefore$ $$I = \frac{1}{2}\int \frac{dt}{t^2+1} = \frac{1}{2}\tan^{-1} t = \frac{1}{2}\tan^{-1}(\tan x)$$

To integrate $\int \frac{a\sin x+b\cos x}{c\sin x+d\cos x}dx$

Put the numerator = A(denominator) + B(Diff. coefficient of denominator).

Example

Solve

(i) $\int \frac{\sin x+8\cos x}{2\sin x+3\cos x}dx$

(ii) $\int \frac{dx}{a+b\tan x}$

Solution

(i) Let, sinx + 8cosx = A(2sinx + 3cosx) + B(2cosx - 3sinx)

Equating coefficients of sinx and cosx, we get,

2A - 3B = 1 or 2A - 3B -1 = 0

and

3A + 2B = 8 or 3A + 2B - 8 = 0

$$\therefore \quad \frac{A}{26} = \frac{B}{13} = \frac{1}{13} \rightarrow A = 2 \text{ and } B = 1$$

sinx + 8cosx = 2(2sinx+3cosx) + 1(2cosx - 3sinx)

$$I = 2\int dx + \int \frac{2\cos x - 3\sin x}{2\sin x + 3\cos x} dx = 2x + \log(2\sin x + 3\cos x)$$

(ii) $$I = \int \frac{dx}{a + b\tan x} = \int \frac{\cos x.dx}{a\cos x + b\sin x} \quad \text{(i)}$$

Now, let cosx = A(acosx + bsinx) + B(-asinx + bcosx) (ii)

Equating the coefficients of cosx and sinx, we have

1 = Aa + Bb (iii)

0 = Ab - Bb (iv)

Solving, we get

$$A = \frac{a^2}{a^2 + b^2} \text{ and } B = \frac{b}{a^2 + b^2}$$

$\therefore$ from (ii), we have

$$\cos x = \frac{a}{a^2 + b^2}(a\cos x + b\sin x) + \frac{b}{a^2 + b^2}(-a\sin x + b\cos x)$$

$\therefore$ From (i), we get

$$I = \int \frac{\frac{a}{a^2 + b^2}(a\cos x + b\sin x) + \frac{b}{a^2 + b^2}(-a\sin x + b\cos x)}{a\cos x + b\sin x}$$

$$= \frac{a^2}{a^2 + b^2}\int dx + \frac{b}{a^2 + b^2}\int \frac{-a\sin x + b\cos x}{a\cos x + b\sin x} dx$$

$$= \frac{ax}{a^2 + b^2} + \frac{b}{a^2 + b^2}\log(a\cos x + b\sin x)$$

Problems

Evaluate

(i) $\sin^5 x \cos^4 x dx$

Ans: $-\cos^5 x/5 + 2\cos^7 x/7 - \cos^9 x/9$

(ii) $\int \frac{dx}{5+4\cos x}$

Ans: $\frac{2}{3}\tan^{-1}\left(\frac{\tan x/2}{3}\right)$

(iii) $\int \frac{dx}{3\cos x - 4\sin x + 5}$

Ans: $\frac{1}{2-\tan(x/2)}$

Hint: Put $\tan(x/2) = t$ *so that* $dx = \frac{2dt}{1+t^2}; \cos x = \frac{1-t^2}{1+t^2}; \sin x = \frac{2t}{1+t^2}$

REDUCTION FORMULA

In this method, we go on reducing the power till we get a power whose integral is known or which can be integrated easily. reduction formula is generally obtained by the method of integration by parts.

Reduction Formula for $\int \sin^n x.dx$

$$I = \int \sin^n x.dx = \int \sin^{n-1} x \sin x dx$$

Now integrating by parts, we get

$$I = \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2} x \cos x(-\cos x)dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)dx$$

$\therefore$ $$I = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x dx - (n-1)\int \sin^n x dx$$

Or $$I + (n-1)I = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x dx$$

Or $$nI = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x dx$$

$\therefore$ $$I = -\frac{\sin^{n-1} x \cos x}{n} + \frac{(n-1)}{n}\int \sin^{n-2} x dx$$

Note: Reduction formula for $\int \cos^n x.dx$ may also be found out by following the above procedure. The result comes as,

$$\int \cos^n x.dx = \frac{\cos^{n-1} x.\sin x}{n} + \frac{(n-1)}{n}\int \cos^{n-2} dx.dx$$

Reduction Formula for $\int \tan^n xdx$

$$I = \int \tan^n dx = \int \tan^{n-2} x \tan^2 xdx$$

$$= \int \tan^{n-2} x(\sec^2 x - 1)dx$$

$$= \int \tan^{n-2} x.\sec^2 x.dx - \int \tan^{n-2} x.dx$$

$$= \frac{(\tan x)^{n-2+1}}{n-2+1} - \int \tan^{n-2} x.dx$$

$$\therefore \qquad I = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x.dx$$

Note: Reduction formula for $\int \cot^n x.dx$ may also be found out by following the above procedure. The result comes as,

$$\int \cot^n x.dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x.dx$$

Reduction Formula for $\int \sec^n x.dx$

$$I = \int \sec^n xdx = \int \sec^{n-2} x.\sec^2 x.dx$$

Integrating by parts, we have

$$I = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-3} x(\sec x \tan x)\tan xdx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)dx$$

$$= \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x - (n-2)\int \sec^n xdx$$

Or $$I = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} xdx - (n-2)I$$

Or $$I(n-1) = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x.dx$$

Or $$I = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x.dx$$

Note:

Reduction formula of $\int \cos ecx^n xdx$ may also be easily found out by following the above procedure. The result comes as,

$$\int \cos ec^n x.dx = -\frac{\cos ec^{n-2} x.\cot x}{n-1} + \frac{n-2}{n-1}\int \cos ec^{n-2} x.dx$$

Example

Evaluate

(i) $\int \sin^6 x.dx$

(ii) $\int \tan^4 x$

Solution

(i) $I = \int \sin^6 xdx = -\frac{\cos x \sin^5 x}{6} + \frac{5}{6}\int \sin^4 xdx$ (By applying reduction formula)

$$= -\frac{\sin^5 x \cos x}{6} + \frac{5}{6}\left[\frac{-\cos x \sin^3 x}{4} + \frac{3}{4}\int \sin^2 x.dx\right]$$

$$= -\frac{\sin^5 x \cos x}{6} - \frac{5}{24}\cos x \sin^3 x + \frac{5}{8}\int \frac{1-\cos 2x}{2} dx$$

$$= -\frac{\sin^5 x \cos x}{6} - \frac{5}{24}\cos x \sin^3 x + \frac{5}{16}x - \frac{5}{32}\sin 2x$$

$$= -\frac{\sin^5 x \cos x}{6} - \frac{5}{24}\cos x \sin^3 x + \frac{5}{16}x - \frac{5}{16}\sin x \cos x$$

(ii) $I = \int \tan^4 x = \frac{\tan^3 x}{3} - \int \tan^2 x.dx$ [By reduction formula]

$$= \frac{\tan^3 x}{3} - \left[\frac{\tan x}{1} - \int \tan^{2-2} xdx\right]$$

$$= \frac{\tan^3 x}{3} - [\tan x - x]$$

Note

Remember the following values:

(i) $\int_0^{\pi/2} \sin^n x.dx = \int_0^{\pi/2} \cos^n x.dx$

$$= \frac{(n-1)(n-3)(n-5).......}{n(n-2)(n-4).......}.\frac{\pi}{2} \quad \text{(when n is even)}$$

$$= \frac{(n-1)(n-3)(n-5).....}{n(n-2)(n-4).....} \quad \text{(No } \pi/2 \text{ when n is odd)}$$

(ii) $\int_0^{\pi/2} \sin^m x \cos^m x.dx = \frac{[(m-1)(m-3)(m-5)...][(n-1)(n-3)]}{(m+n)(m+n-2)(m+n-4)......}.\frac{\pi}{2}$

(If "n" is even)

=

$$\int_0^{\pi/2} \sin^m x \cos^n x.dx = \frac{[(m-1)(m-3)(m-5)....][(n-1)(n-3)(n-5)]}{(m+n)(m+n-2)(m+n-4)}$$

(If "n" is odd)

Reduction Formula for $\int \sin^m x.\cos^n x.dx$

$$\int \sin^n x \cos^n xdx = \int [\sin^n x \cos^{n-1} x] \cos x.dx$$

Integrating by parts, we have

$$I = \sin^m x \cos^{n-1} x(\sin x) - \int \{m \sin^{m-1} x \cos x \cos^{n-1} x + \sin^m x(n-1)\cos^{n-2} x(-\sin x)\} \sin xdx$$

$$\therefore \quad I = \sin^{m+1} x \cos^{n-1} x - \int [m \sin^m x \cos^n x - (n-1)\sin^{m+2} x \cos^{n-2} x]dx$$

$$= \sin^{m+1} x \cos^{n-1} x - m\int \sin^m x \cos^n xdx + (n-1)\int \sin^m x \cos^{n-1} x(1-\cos^2 x)dx$$

$$= \sin^{m+1} x \cos^{n-1} x - m\int \sin^m x \cos^n x.dx + (n-1)\int \sin^m x \cos^{n-2} x - (n-1)\int \sin^m x \cos^n xdx$$

$$\therefore \ (m+n)I = \sin^{m+n} x.\cos^{n-1} x + (n-1)\int \sin^m x \cos^{n-2} xdx$$

$$\therefore \ I = \int \sin^m x \cos^n xdx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x.\cos^{n-2} x.dx$$

Reduction Formula for $\int \cos^m x \cos nxdx$

In integrating by parts, we have

$$I = \int \cos^m x \cos nxdx = \frac{\cos^m x \sin nx}{n} - \int m\cos^{m-1} x(-\sin x).\frac{\sin nx}{n}.dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{n-1} x \sin nx.dx$$

We know that

$$\cos(n-1)x = \cos nx \cos x - \sin nx \sin x$$

$$\therefore \quad \sin nx \sin x = \cos(n-1)x - \cos nx \cos x$$

$$\therefore \quad \int \cos^m x \cos nx.dx = \frac{\cos^m x \sin x}{n} + \frac{m}{n}\int \cos^{m-1} x\{\cos(n-1)x - \cos nx \cos x\}dx$$

$$= \frac{\cos^n x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \cos(n-1)xdx - \frac{m}{n}\int \cos^m x \cos nxdx$$

$$\therefore \left(1+\frac{m}{n}\right)\int \cos^n x \cos nx.dx = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \cos(n-1)x.dx$$

$$\therefore \int \cos^m x \cos nx.dx = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n}\int \cos^{m-1} x \cos(n-1)x.dx$$

Or $\quad I_{m,n} = \frac{\cos^m x \sin x}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$

Example

Solve

(i) $\quad \int_0^{\pi/2} \sin^4 xdx = \frac{3x1}{4x2}.\frac{\pi}{2} = \frac{3\pi}{16}$

(ii) $\quad \int_0^{\pi/2} \sin^6 x \cos^4 xdx = \frac{(5x3x1)(3x1)}{10x8x6x4x2}.\frac{\pi}{2} = \frac{3\pi}{512}$

Example

Evaluate $\int_0^4 x^3\sqrt{4x - x^2}.dx$

Solution

$$I = \int_0^4 x^3\sqrt{4x-x^2}.dx = \int_0^4 x^{7/2}\sqrt{4-x}.dx$$

Put, $\quad x = 4\sin^2\theta \qquad \therefore \; dx = 4(2\sin\theta\cos\theta)d\theta = 8\sin\theta.\cos\theta.d\theta$

$$\therefore \qquad I = \int_0^{\pi/2} (4\sin^2\theta)^{7/2}\sqrt{4-4\sin^2\theta}.(8\sin\theta.\cos\theta)d\theta$$

$$= (4)^{7/2}.16\int_0^{\pi/2} \sin^8\theta\cos^2\theta.d\theta$$

$$= (2)^7(2)^4.\frac{(7x5x3x1)x1}{10x8x6x4x2}.\frac{\pi}{2} = 56\pi$$

Example

If $I = \int_0^{\pi/2} x^n \sin x.dx$ and $n > 1$, *show that*

$$I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$

Solution

Integrating by parts, we have

$$I = x^n(-\cos x) - \int nx^{n-1}(-\cos x)dx$$

$$= -x^n \cos x + n\int x^{n-1}\cos x.dx$$

$$\therefore \quad I_n = \left[-x^n \cos x\right]_0^{\pi/2} + n\int_0^{\pi/2} x^{n-1}.dx$$

$$= 0 + n\left[\left(x^{n-1}\sin x\right)_0^{\pi/2} - \int_0^{\pi/2}(n-2)x^{n-2}\sin x dx\right]$$

$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\int_0^{\pi/2} x^{n-2}.\sin x.dx$$

$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$$

$$\therefore \quad I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$

Problem

If $I_n = \int_0^{\pi/2} x.\text{cox}^n x.dx$, *prove that*

$I_n = \frac{n-1}{n}I_{n-2} - \frac{1}{n^2}$. *Hence evaluate* I_4.

Answer: $\frac{3\pi^2}{64} - \frac{1}{4}$

$$I_4 = -\frac{1}{4^2} + \frac{3}{4}.I_2$$

Hint: $I_2 = -\frac{1}{2^2} + \frac{1}{2}I_0$

$$I_0 = \int_0^{\pi/2} x.dx = \frac{\pi^2}{8}$$

DEFINITE INTEGRALS

If $\frac{d}{dx}F(x) = f(x)$, then the definite integral of f(x) between the limits x = a and x = b is defined as $\int_a^b f(x)dx = |F(x)|_a^b = F(b) - F(a)$

Example

$$\int_1^2 x.dx = \left(\frac{x^2}{2}\right)_1^2 = \frac{2^2}{2} - \frac{1^2}{2} = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$$

Properties of Definite Integrals

S.No.	Property	Explanation/Example
1.	$\int_a^b f(x)dx = -\int_b^a f(x).dx$	
2.	$\int_a^b f(x).dx = -\int_c^d f(t).dt$	That is, limits change accordingly with the substitute "t", which is substituted in place of "x". Also, "dt" replaces "dx". The value of "dt" is obtained by differentiating "t" w.r.t. "x".
3.	$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,\ \text{if } a < c < b$	
4.	$\int_0^a f(x).dx = \int_0^a f(a-x).dx$	$\int_0^{\pi/2} \sin^n x.dx$ $= \int_0^{\pi/2} \sin^n\left(\frac{\pi}{2} - x\right).dx$ $= \int_0^{\pi/2} \cos^n xdx$
5.	$\int_{-a}^a f(x)dx = 0$, when f(x) is odd function $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$, when f(x) is an even function.	(i) f(x) is an odd function if f(-x) = -f(x). For example, sin(-x) = -sinx, hence sinx is odd function. (ii) f(x) is an even function if f(-x) = f(x). For example, cos(-x) = cosx, hence cosx is an even function.
	$\int_0^{2a} f(x).dx = 2\int_0^a f(x).dx$, if f(2a-x) = f(x) $\int_0^{2a} f(x).dx = 0$, if f(2a-x) = -f(x)	

Example

Evaluate

(i) $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}.dx$

(ii) $\int_0^{\pi/2} \log(\sin x)dx$ *[Very Important]*

(iii) $\int_0^{\pi} \frac{x.dx}{a^2 - \cos^2 x}$

Solution

(i) $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$

$$= \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)dx}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}$$ [Using property 4]

$$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx$$

Hence

$$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}.dx = \int_0^{\pi/2} dx = |x|_0^{\pi/2} = \frac{\pi}{2}$$

(ii) $I = \int_0^{\pi/2} \log(\sin x)$

$$= \int_0^{\pi/2} \log\sin\left(\frac{\pi}{2} - x\right).dx$$ [Using property 4]

$$= \int_0^{\pi/2} \log\cos x.dx$$

Hence

$$2I = \int_0^{\pi/2} (\log\sin x + \log\cos x).dx$$

$$= \int_0^{\pi/2} \log\sin x.\cos x.dx$$

$$= \int_0^{\pi/2} \log\left(\frac{2\sin x.\cos x}{2}\right)dx$$

$$= \int_0^{\pi/2} \log.\sin 2x - \int_0^{\pi/2} \log 2.dx$$

$$= I_1 - \log 2\int_0^{\pi/2} dx$$

$$= I_1 - \log 2.|x|_0^{\pi/2}$$

$$= I_1 - \frac{\pi}{2}\log 2$$

Let us find $I_1 = \int_0^{\pi/2} \log\sin 2x.dx$

Put 2x = t $\quad \therefore$ 2dx = dt $\quad$ or dx = dt/2

When x = 0, t = 0

When x = π/2, t = π

Hence

$$I_1 = \frac{1}{2}\int_0^{\pi} \log\sin t.dt = \frac{1}{2}.2\int_0^{\pi/2} \log.\sin t.dt \qquad \text{[Using property 2]}$$

$$= \int_0^{\pi/2} \log.\sin t.dt = \int_0^{\pi/2} \log.\sin x.dx = I$$

Hence 2I = I - (π/2).log2

or $\quad$ I = -(π/2) log2

(iii) $$I = \int_0^{\pi} \frac{x.dx}{a^2 - \cos^2 x} = \int_0^{\pi} \frac{(\pi - x)dx}{a^2 - \cos^2(\pi - x)}$$

$$= \int_0^{\pi} \frac{\pi.dx}{a^2 - \cos^2 x} - I$$

$$2I = \int_0^{\pi} \frac{\pi.dx}{a^2 - \cos^2 x} = \int_0^{\pi} \frac{\pi.dx}{a^2 - \cos^2 x}$$

Dividing numerator and denominator by $\cos^2 x$, we get

$$I = \pi\int_0^{\pi/2} \frac{\sec^2 x.dx}{a^2(1 + \tan^2 x) - 1} = \pi\int_0^{\pi/2} \frac{\sec^2 xdx}{(a^2 - 1)^2 + a^2\tan^2 x}$$

Put $\quad$ atanx = t $\quad \therefore$ $\sec^2 x.dx = dt/a$

when $\quad$ x = 0, t = 0

when $\quad$ x = π/2, t = ∞

$$\therefore \quad I = \frac{\pi}{a}\int_0^{\infty} \frac{dt}{(a^2 - 1) + t^2} = \frac{\pi}{a}.\frac{1}{\sqrt{a^2 - 1}}\left(\tan^{-1}\left\{\frac{t}{\sqrt{a^2 - 1}}\right\}\right)_0^{\infty}$$

$$= \frac{\pi}{a\sqrt{a^2-1}}\left[\tan^{-1}\infty - \tan^{-1}0\right] = \frac{\pi}{a\sqrt{a^2-1}}.\frac{\pi}{2}$$

$$= \frac{\pi^2}{2a\sqrt{a^2-1}}$$

Example

Show that $\int_0^{\pi} xf(\sin x)dx = \pi\int_0^{\pi/2} f(\sin x)dx$

Solution

Let $\quad I = \int_0^{\pi} xf(\sin x)dx \qquad (1)$

Then $\quad I = \int_0^{\pi}(\pi - x)f[\sin(\pi - x)]dx = \int_0^{\pi}(\pi - x)f(\sin x)dx \qquad (2)$

Adding (1) and (2)

$$2I = \int_0^{\pi} \pi f(\sin x)dx$$

or $\quad I = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$

Here $\quad f(\sin x) = f[\sin(\pi-x)]$

$\therefore \quad I = 2.\frac{\pi}{2}\int_0^{\pi/2} f(\sin x)dx = \pi\int_0^{\pi/2} f(\sin x)dx$

ASSIGNMENT

Q.1. (AMIE S09, 11, W09, 8 marks): If $I_{m,n} = \int \cos^m x \sin nx dx$, prove that

$$I_{m.n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

Hence or otherwise, evaluate

$$\int_0^{\pi/2} \cos^5 x \sin 3x dx$$

Q.2. (AMIE W09, 4 marks): Obtain the reduction formula for $\int_0^1 x^m (1-x)^n dx$

Answer: $I_{mn} = \frac{x^{m+1}}{m+n+1}(1-x)^n + \frac{n}{m+n+1} I_{m,n-1}$

Q.3. (AMIE W09, 8 marks): Verify whether the two integrals

$$I_1 = \int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3} dy$$

and $I_2 = \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3} dx$ are equal.

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